

# A GENERALIZATION OF THE GAUSS-BONNET AND HOPF-POINCARÉ THEOREMS. PART II

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**ABSTRACT.** This paper is a continuation of [1]. Let  $\pi : E \rightarrow M$  be a locally trivial fiber bundle over a two-dimensional manifold  $M$ , and  $\Sigma \subset M$  be a discrete subset. A subset  $Q \subset E$  is called a *n-sheeted branched section of the bundle  $\pi$*  if  $Q' = \pi^{-1}(M \setminus \Sigma) \cap Q$  is a *n-sheeted covering of  $M \setminus \Sigma$* . The set  $\Sigma$  is called the *singularity set* of the branched section  $Q$ . We define the index of a singularity point of a branched section, and give examples of its calculation, in particular for branched sections of the projective tangent bundle of  $M$  determined by binary differential equations. Also we define a resolution of singularities of a branched section, and prove an analog of Hopf-Poincaré-Gauss-Bonnet theorem for the branched sections admitting a resolution.

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## 1. INTRODUCTION

Let us recall that a branched covering is a smooth map  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are compact  $n$ -dimensional manifolds, such that  $df_x : T_x X \rightarrow T_{f(x)} Y$  is an isomorphism for all points  $x \in X \setminus A$  for some subset  $A \subset X$  of dimension less or equal to  $n - 2$ . In this case, if  $X' = X \setminus f^{-1}(f(A))$  and  $Y' = Y \setminus f(A)$ , then the induced map  $f' : X' \rightarrow Y'$  is a finite-sheeted covering map. The points of the set  $f(A)$  are called the *branch points* of the branch covering  $f$  ([2], Section 18.3).

Now let  $\xi = \{\pi_E : E \rightarrow M\}$  be a fiber bundle. Let  $\Sigma$  be a closed subset of  $M$ ,  $M' = M \setminus \Sigma$ , and  $E' = \pi^{-1}(M')$ .

**Definition 1.** An *n-sheeted branched section* of the bundle  $\xi$  is a subset  $Q \subset E$  such that  $Q' = Q \cap E'$  is an embedded submanifold of  $E$  and  $\pi_E|_{Q'} : Q' \rightarrow M'$  is a *n-sheeted covering*. The set  $\Sigma$  is called the *singularity set* of the branched section  $Q$ .

*Example 1.* Let  $V$  be a section of the tangent bundle  $\pi_{TN} : TN \rightarrow N$ , and  $f : N \rightarrow M$  be a *k-sheeted covering*, then we can construct a branched section  $df(V)$  of the tangent bundle  $\pi_{TM} : TM \rightarrow M$  in the following way. Let us consider the subset  $Q = \{df_y(V(y)) \mid y \in N\} \subset TM$ . For each  $x \in M$ , let us set  $\mathcal{V}(x) = \{df_y(V(y)) \mid y \in f^{-1}(x)\} \subset T_x M$ . Take the subset  $\Sigma \subset M$  consisting of points  $x \in M$  such that the number of elements of the set  $\mathcal{V}(x)$  is less than  $k$ . Then  $M' = M \setminus \Sigma$  is open,  $Q' = Q \cap \pi_{TM}^{-1}(M')$  is a submanifold of  $TM$  and  $f$  induces a *k-sheeted covering*  $f' : Q' \rightarrow M'$ . Indeed, for each  $x \in M'$  there exists a neighborhood  $U \subset M'$  of  $x$  such that  $f^{-1}(U) = \bigsqcup_{j=1}^k \tilde{U}_j \subset N$  and, for each  $j = \overline{1, k}$ , the application  $f_j = f|_{\tilde{U}_j} : \tilde{U}_j \rightarrow U$  is a diffeomorphism. Therefore  $df_j : T\tilde{U}_j \subset TN \rightarrow TU \subset TM$  is also a diffeomorphism. As  $V : \tilde{U}_j \rightarrow V(\tilde{U}_j) \subset TN$  is a diffeomorphism onto its image,

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the map  $\theta_j = df_j \circ V \circ f_j^{-1} : U \rightarrow df_j(V(\tilde{U}_j)) \subset Q \subset TM$ ,  $j = \overline{1, k}$ , is a diffeomorphism onto its image, as well. Note that, for each  $y \in U \subset M'$ , we have that the set  $f^{-1}(y) = \{p_j \in \tilde{U}_j\}_{j=\overline{1, k}}$  consists of  $k$  distinct points, and the set  $\{df_{p_j}(V(p_j))\}_{j=\overline{1, k}}$  consists of  $k$  distinct vectors, by the definition of  $M'$ . Therefore,  $\theta_i(U) \cap \theta_j(U) = \emptyset$ , for  $i \neq j$ . Thus  $\pi_{TM}^{-1}(U) \cap Q' = \bigsqcup_{j=1}^k \theta_j(U)$ , this means that  $U$  is simply covered in  $Q'$ , and  $Q'$  is a  $k$ -sheeted covering of  $M'$ .

The branched sections naturally appear in the theory of differential equations over manifolds. Our main example in this paper is the following one.

*Example 2.* Let  $M$  be a connected compact oriented manifold and let  $\omega$  be a symmetric tensor of order  $n$  over  $M$ . Recall that such a tensor can be written locally as follows

$$(1) \quad \omega_{(x,y)} = a_0(x,y)dx^n + a_1(x,y)dx^{n-1}dy + \cdots + a_n(x,y)dy^n,$$

where  $(x,y)$  are coordinate functions on an open set  $U \subset M$ , and  $a_i : U \rightarrow \mathbb{R}$  are smooth functions defined in  $U$ . In what follows, we suppose that  $\omega$  has the following properties:

- (1) The function  $\omega_{(x,y)}$  is identically zero if and only if  $a_i(x,y) = 0$  for  $0 \leq i \leq n$ . We set  $\Sigma = \{p \in M : \omega_p = 0\}$ .
- (2) On  $M \setminus \Sigma$ , the tensor  $\omega$  has the form  $\omega = \lambda_1 \lambda_2 \cdots \lambda_n$ , where  $\lambda_i \in \Omega(M \setminus \Sigma)$  pairwise linearly independent.

*Statement 1.* The  $n$ -form  $\omega$  determines a branched section of the bundle  $\pi : PTM \rightarrow M$

*Proof.* Let  $Q$  be the solution on  $PTM$  of the equation (1). We will prove that  $Q$  is a branched section of  $\pi$ . Let  $E' = \pi^{-1}(M \setminus \Sigma)$  and  $Q' = Q \cap E'$ . It follows from the property (2) that the set  $F_p = Q \cap \pi^{-1}(p)$ ,  $p \in M \setminus \Sigma$  has exactly  $n$  elements, therefore each fiber of the surjective map  $\pi' := \pi|_{Q'} : Q' \rightarrow M \setminus \Sigma$  is finite with  $n$  elements. On the other hand, if  $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}P^1$  is a trivialization of  $PTM$  on  $U$ , then the restriction  $\varphi' := \varphi|_{\pi^{-1}(U) \cap Q'} : \pi^{-1}(U) \cap Q' \rightarrow U \times \mathbb{R}P^1$  is a homeomorphism on its image. Since  $\pi|_{\pi^{-1}(U) \cap Q'} : \pi^{-1}(U) \cap Q' \rightarrow U \cap (M \setminus \Sigma)$  has finite fiber with  $n$  elements over each point  $p \in U \cap (M \setminus \Sigma)$ , from the following commutative diagram

$$(2) \quad \begin{array}{ccc} & \varphi'(\pi^{-1}(U \cap (M \setminus \Sigma)) \cap Q') & \\ & \swarrow pr_1 & \uparrow \varphi' \\ U \cap (M \setminus \Sigma) & \xleftarrow{\pi'} \pi^{-1}(U \cap (M \setminus \Sigma)) \cap Q' & \end{array}$$

It follows that  $\pi|_{Q'} : Q' \rightarrow M \setminus \Sigma$  is a local diffeomorphism. Therefore,  $\pi|_{Q'} : Q' \rightarrow M \setminus \Sigma$  is a  $n$ -sheeted branched covering, and so  $Q$  is a branched section of  $PTM$ .  $\square$

Geometrically  $Q$  determines an  $n$ -web at the points of  $M \setminus \Sigma$ .

*Example 3.* Let  $\xi = \{\pi : \overline{P} \rightarrow M\}$  be a  $\overline{G}$ -principal bundle which reduces to a finite subgroup  $G \subset \overline{G}$  over  $M \setminus \Sigma$ , where  $\Sigma \subset M$  is a closed subset. Then the corresponding  $G$ -principal bundle  $P \subset \overline{P}$  is a branched section of the bundle  $\xi$  with singularity set  $\Sigma$ .

For example, let  $M$  be a two-dimensional oriented Riemannian manifold, and  $\overline{P} = SO(M)$ , the  $SO(2)$ -principal bundle of orthonormal positively oriented frames of  $M$ . Any finite subgroup  $G \subset SO(2)$  is a cyclic group  $G \cong \mathbb{Z}_m$  generated by the rotation  $R_{2\pi/m}$ .

If  $P \subset SO(M') \subset SO(M)$  is a reduction of  $SO(M)$  to  $G$  over  $M' = M \setminus \Sigma$ , then at each point  $x \in M'$  we have the set  $\mathcal{N}(x) = \{e \in T_x M \mid (e, R_{\pi/2} e) \in P\}$ , which consists of  $m$  unit vectors such that the angle between any two of them is  $2\pi l/m$ . The set  $\mathcal{N}(x)$  defines a regular  $m$ -polygon  $P_m \subset T_x M'$  inscribed in a unit circle centered at  $0 \in T_x M$ .

It is clear that, vice versa, if at any point of  $M' = M \setminus \Sigma$ , we are given a unitary  $m$ -polygon  $P_m \subset T_p M'$  and the field of these polygons is smooth (these means that locally we can choose  $m$  unitary vector fields whose values are the vertices of the polygons  $P_m$ ), then the bundle  $SO(M)$  reduces to the subgroup  $G \cong \mathbb{Z}_m$  of the Lie group  $SO(2)$ .

This situation occurs, for example, when  $M$  is a surface in  $\mathbb{R}^3$ , and  $\Sigma$  is the set of umbilic points of  $M$ . Then at each point of  $M'$  we have two orthogonal eigenspaces of the shape operator of the surface, which determine a square in  $T_p M$  with vertices at points where these eigenspaces meet the unit circle centered at  $0 \in T_p M$ . Therefore, over  $M' = M \setminus \Sigma$  the bundle  $SO(M)$  reduces to the subgroup  $G \cong \mathbb{Z}_4$  generated by the rotation  $R_{\pi/2}$ . The corresponding principal subbundle  $P$ , the branched section of the bundle  $SO(M) \rightarrow M$ , consists of oriented orthogonal frames such that the frame vectors span the eigenspaces.

Moreover, as the difference of the principal curvatures never vanish on  $M'$ , we can order the principal curvatures in such a way that  $k_1(p) > k_2(p)$  at any  $p \in M'$ . Let  $L_a(p)$ ,  $a = 1, 2$  be the eigenspace corresponding to the principal curvature  $k_a(p)$ ,  $a = 1, 2$ . Then we can choose the subbundle  $P \subset SO(M)$  in such a way that, for  $\{e_1, e_2\} \in P$ , the vector  $e_a$  spans  $L_a$ ,  $a = 1, 2$ , therefore in this case the bundle  $SO(M) \rightarrow M$  reduces to the group  $G \cong \mathbb{Z}_2$ .

Also, note that this example is related to Example 2. Indeed if we have the reduction of  $P \subset SO(M)$  to the subgroup  $G \cong \mathbb{Z}_m$  over  $M'$ , then at each point  $p \in M'$  we have  $m$  (or  $m/2$ ) subspaces spanned by the vector  $e_1$  from the frame  $\{e_1, e_2\} \in P$ . Then we can take the binary differential equation (1) such that these subspaces are the roots of the corresponding algebraic equation.

The paper is organized as follows. In Section 2 we define the index of an isolated singular point of a branched section of locally trivial bundle  $\xi = \{\pi_E : E \rightarrow M\}$  over a two-dimensional oriented manifold  $M$  (see Definition 2), this definition generalizes the definition of the index of a singular point of a section from ([1], Section 2.2, Definition 1). In Section 3 we define a resolution of a branched section (see Definition 3), and give various examples of resolutions (see Examples 6 – 9). And, finally, in Section 3 we prove an analogue of the Gauss-Bonnet theorem for a branched section which admits resolution (see Theorem 1).

## 2. THE INDEX OF A SINGULAR ISOLATED POINT

**2.1. Local monodromy group.** Let  $M$  be a two-dimensional closed oriented manifold. Let  $\xi = \{\pi_E : E \rightarrow M\}$  be a fiber bundle with oriented typical fiber  $F$ .

Let us consider a  $k$ -sheeted branched section  $Q$  of  $\xi$  (see Definition 1) with singularity set  $\Sigma$ , and let  $\pi_Q = \pi_E|_Q : Q \rightarrow M$ . Recall that  $M' = M \setminus \Sigma$ ,  $E' = \pi^{-1}(M')$ , and  $Q' = Q \cap E'$ .

Assume that  $x \in \Sigma$  is an isolated point of  $\Sigma$ . Let us take a neighborhood  $U(x)$  such that  $U'(x) = U(x) \setminus \{x\}$  is an open subset of  $M'$  and there exists a diffeomorphism  $\varphi : (D, 0) \rightarrow (U(x), x)$ , where  $D \subset \mathbb{R}^2$  is the standard open 2-disk centered at the origin  $0 \in \mathbb{R}^2$ . We will call  $U(x)$  a *disk neighborhood* of  $x$  and assume that  $\varphi$  sends the standard orientation of  $D$  to the orientation of  $U(x)$  induced by the orientation of  $M$ .

By Definition 1, the map  $\pi_Q|_{\pi_Q^{-1}(U'(x))} : \pi_Q^{-1}(U'(x)) \rightarrow U'(x)$  is a  $k$ -sheeted covering.

If  $U(x)$  is a disk neighborhood of an isolated point  $x \in \Sigma$ , then for each point  $y \in U'(x)$ , the fundamental group  $\Pi_1(y) = \pi_1(U'(x), y)$  is isomorphic to  $\mathbb{Z}$ . There are two generators of  $\Pi_1(y)$ :  $[\gamma_+]$  and  $[\gamma_-]$ , where  $\gamma_{\pm} = \phi(C_{\pm})$  and  $C_{\pm}$  is a circle in  $D$  passing through the point  $\varphi^{-1}(y)$  and enclosing the origin, and having positive (negative, respectively) orientation. We will call the element  $[\gamma_{\pm}] \in \Pi_1(y)$  the positive (the negative, respectively) generator of  $\Pi_1(y)$ .

The group  $\Pi_1(y)$  acts on the fiber  $Q_y = \pi_Q^{-1}(y)$  in the following way: for an element  $[\gamma] \in \Pi_1(y)$  and  $q \in Q_y$  we set  $[\gamma] \cdot q = \bar{q}$  if the lift  $\tilde{\gamma}$  of  $\gamma$  starting at  $q$  terminates in  $\bar{q}$ . This action is well defined, this means that if  $\gamma_1$  and  $\gamma_2$  represent the same element in  $\Pi_1(y)$ , then the lifts  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  starting at a same point  $q$  terminate at a same point  $\bar{q}$ .

This action is a homomorphism of the group  $\Pi_1(y)$  to the group of permutations of the fiber  $Q_y$  and its image is called the *local monodromy group* of the branched section  $Q$  at the point  $y \in M'$ .

**Statement 2.** *The local monodromy group does not depend on a choice of the disk neighborhood  $U(x)$ .*

*Proof.* Let  $U(x)$  and  $V(x)$  be two disk neighborhoods of  $x$ , and  $y$  lies in  $U(x) \cap V(x)$ . Then  $\Pi_1^U(y) = \pi_1(U'(x), y) = \Pi_1^V(y) = \pi_1(V'(x), y)$  because for each class  $[\gamma] \in \pi_1(V'(x), y)$  or  $[\gamma] \in \pi_1(U'(x), y)$  one can find a representative  $\gamma_1 \in [\gamma]$  which takes values in  $U'(x) \cap V'(x)$ .  $\square$

**Statement 3.** *Let  $\gamma$  be a loop in  $U'(x)$  based at a point  $y \in U'(x)$  such that its homotopic class represents the positive generator of  $\Pi_1(y)$ . Then for each orbit  $O$  of the local monodromy group action on  $Q_y$  and each point  $q \in O$ , there exists a loop  $\tilde{\gamma}$  in  $\pi_Q^{-1}(U'(x))$  based at  $q$  which passes through each point of the orbit once and only once and such that  $\pi_1(\pi_E)([\tilde{\gamma}]) = [\gamma]^k$ , where  $k$  is the number  $\#O$  of elements of the orbit  $O$ . Here  $\pi_1(\pi_E) : \pi_1(\pi_Q^{-1}(U'(x)), q) \rightarrow \pi_1(U(x), y)$  is the homomorphism of the fundamental groups induced by the map  $\pi_E$ .*

*Proof.* First of all note that if we have an action of the group  $\mathbb{Z}$  on a finite set, then we can enumerate elements of each orbit in such a way that the action of the group generator 1 on this orbit is represented by the cycle  $\sigma = (2, 3, \dots, 1)$ . Indeed, let  $O$  be an orbit of the action, and  $q \in O$ . The map  $F : \mathbb{Z}/H_q \rightarrow O$ ,  $[g] \rightarrow g \cdot q$ , where  $H_q$  is the isotropy subgroup of the action, is an equivariant bijection. The group  $H_q$  is a cyclic group, this means that there exists  $k \in \mathbb{Z}$ ,  $k \geq 0$  such that  $H_q = \{km \mid m \in \mathbb{Z}\}$ , therefore  $\mathbb{Z}/H_q = \{[0], [1], \dots, [k-1]\}$ , and the action of the generator  $1 \in \mathbb{Z}$  on  $\mathbb{Z}/H_q$  is given exactly by the cycle  $\sigma$ .

Now, for a point  $y \in U'(x)$ , let  $[\gamma]$ ,  $\gamma : [0, 1] \rightarrow U'(x)$ , be the positive generator of  $\Pi_1(y)$ . Let us take an orbit  $O$  of the local monodromy group action on  $Q_y$  and a point  $q \in O$ . Let  $k$  be the number of elements of  $O$ . As we have seen, the action of  $[\gamma]$  on  $O$  is represented by the cycle  $\sigma$ , this means we can enumerate the points of the orbit  $O$  in such a way that  $q_1 = q$ ,  $[\gamma]q_1 = q_2$ ,  $\dots$ ,  $[\gamma]q_{k-1} = q_k$ , and  $[\gamma]q_k = q_1 = q$ . Therefore, by the construction of the action of  $\Pi_1(y)$  on  $Q_y$ , for the lift  $\tilde{\gamma}_1$  of  $\gamma$  to  $Q'$  such that  $\tilde{\gamma}_1(0) = q_1$  we have that  $\tilde{\gamma}_1(1) = q_2$ , for the lift  $\tilde{\gamma}_2$  of  $\gamma$  to  $Q'$  such that  $\tilde{\gamma}_2(0) = q_2$  we have that  $\tilde{\gamma}_2(1) = q_3$ ,  $\dots$ , and finally for the lift  $\tilde{\gamma}_k$  of  $\gamma$  to  $Q'$  such that  $\tilde{\gamma}_k(0) = q_k$  we have that  $\tilde{\gamma}_k(1) = q_1 = q$ .

What do we do in fact is that we take a point  $q_1 = q \in Q_y$ , then construct the points  $q_2 = [\gamma]q_1$ ,  $q_3 = [\gamma]q_2$ ,  $\dots$ , up to  $[\gamma]q_k = q_1$ . Then the set  $\{q_1, q_2, \dots, q_k\}$  is the orbit  $O$  of the point  $q$ .

It is clear that  $\tilde{\gamma} = \tilde{\gamma}_k \cdot \tilde{\gamma}_{k-1} \cdot \dots \cdot \tilde{\gamma}_1$ , where  $\cdot$  is the path composition, is a loop in  $\pi_Q^{-1}(U'(x))$  at the point  $q_1 = q$ ,  $\tilde{\gamma}$  passes once and only once through each point of  $O$ , and  $\pi_1(\pi_E)([\tilde{\gamma}]) = [\gamma]^k$ . Thus  $\tilde{\gamma}$  is the required loop.  $\square$

**2.2. The index of isolated singular point.** Let  $M$  be an oriented two-dimensional manifold. Let  $\xi = \{\pi_E : E \rightarrow M\}$  be a locally trivial fiber bundle with standard fiber  $F$  and a connected structure Lie group  $G$ .

Let  $Q$  be a branched section of  $\xi$  with singularity set  $\Sigma$ , and  $x$  be an isolated point of  $\Sigma$ . Take a disk neighborhood  $U(x)$ , and for a point  $y \in U'(x)$ , let  $\mathcal{O}_y$  be the set of orbits of local monodromy group action on  $Q_y$ . Take an orbit  $O \in \mathcal{O}(y)$  and a point  $q \in O$ . Let  $[\gamma]$  be a positive generator of the group  $\Pi_1(y)$ , and  $\tilde{\gamma}$  the loop at  $q$  constructed in Statement 3.

Let  $\psi : \pi_E^{-1}(U(x)) \rightarrow U(x) \times F$  be a trivialization of the bundle  $\xi$ , and  $p : \pi_E^{-1}(U(x)) \rightarrow F$  be the corresponding projection. Then the element  $[p \circ \tilde{\gamma}] \in \pi_1(F)$  is called the *index of the branched section  $Q$  at the singular point  $x$  corresponding to the orbit  $O \in \mathcal{O}_y$* , call it  $ind_x(Q; y, O)$ .

**Statement 4.**

- a) The index  $ind_x(Q; y, O)$  does not depend on a choice of the loop  $\gamma : [0, 1] \rightarrow U(x')$  representing the positive generator of the group  $\Pi_1(y)$ .
- b) The index  $ind_x(Q; y, O)$  does not depend on a trivialization.
- c) The index  $ind_x(Q; y, O)$  does not depend on a choice of the disk neighborhood  $U(x)$ , this means that, if  $U(x)$  and  $V(x)$  are two disk neighborhoods of  $x$ , and  $y \in U(x) \cap V(x)$ , then the constructions of  $ind_x(Q; y, O)$  performed for  $U(x)$  and for  $V(x)$  result in the same element in  $\pi_1(F)$ .

*Proof.* a) If  $\gamma$  and  $\mu$  are two representatives of the positive generator of  $\Pi_1(y)$ , then  $\gamma$  is homotopic to  $\mu$ , therefore  $\gamma^k$  is homotopic to  $\mu^k$ , therefore the lift  $\tilde{\mu}$  of  $\mu^k$  is homotopic to the lift  $\tilde{\gamma}$  of  $\gamma^k$ , hence  $p\tilde{\gamma}$  is homotopic to  $p\tilde{\mu}$ .

b) This is because the gluing functions are homotopic to the identity as the structure group is connected.

c) This follows directly from the fact that  $\Pi_1^U(y) = \Pi_1^V(y)$  (see the proof of Statement 3), and from a).  $\square$

*Example 4.* Let us consider the trivial bundle  $\xi = \{\pi_E : E = \mathbb{C} \times \mathbb{C}^* \rightarrow M = \mathbb{C}\}$ , where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $\pi_E(z, w) = z$ . Let us take the subset  $Q = \{(z, w) \mid z^2 = w^3\} \subset E$ .

As  $\pi_E|_Q : Q \rightarrow M \setminus \{z = 0\}$  is a 3-sheeted covering, we see that  $Q$  is a 3-sheeted branched section of the bundle  $\xi$ .

It is clear that the singularity set of  $Q$  is  $\Sigma = \{0\}$ , so  $Q$  has only one singular point  $z = 0$  and this point is isolated. For the disk neighborhood of the isolated singular point  $z = 0$  we take the entire  $M = \mathbb{C}$ .

Let us take  $y = 1$ , then  $Q_y = \{a = (1, 1), b = (1, \varepsilon), c = (1, \varepsilon^2)\}$ , where  $\varepsilon = \exp(2\pi i/3)$ . The loop  $\gamma(t) = \exp(2\pi it)$ ,  $t \in [0, 1]$ , represents the positive generator of the group  $\Pi(y = 1)$ , and the lift  $\tilde{\gamma}_a$  of  $\gamma$  which starts at the point  $a = (1, 1)$  is given by  $\tilde{\gamma}_a(t) = (\exp(2\pi it), \exp(\frac{4}{3}\pi it))$ ,  $t \in [0, 1]$ . Therefore  $[\gamma]a = c$ . In the same manner one can prove that  $[\gamma]b = a$ ,  $[\gamma]c = b$ .

Thus, the orbit  $O$  of the point  $a$  is  $Q_{y=1} = \{a, b, c\}$ , and for a representative of the class  $[\tilde{\gamma}]$  constructed in Statement 3 we can take the loop  $\tilde{\gamma} = (\exp(6\pi it), \exp(4\pi it))$  for  $t \in [0, 1]$ .

Therefore, the loop  $p\tilde{\gamma} : [0, 1] \rightarrow \mathbb{C}^*$  is given by  $p\tilde{\gamma} = \exp(4\pi it)$  for  $t \in [0, 1]$ . Hence

$$(3) \quad ind_0(Q; y = 1, Q_{y=1}) = 2 \in \mathbb{Z} = \pi_1(\mathbb{C}^*).$$

Let us consider the finite set of elements of  $\pi_1(F)$ :

$$(4) \quad \text{ind}_x(Q; y) = \{\text{ind}_x(Q; y, O) \mid O \in \mathcal{O}(y)\}.$$

**Statement 5.** *The set  $\text{ind}_x(Q; y)$  does not depend on  $y \in U'(x)$ .*

*Proof.* Let  $y, \bar{y}$  be two points in  $U'(x)$ . Take a curve  $\delta : [0, 1] \rightarrow U'(x)$  such that  $\delta(0) = y$ ,  $\delta(1) = \bar{y}$ .

The curve  $\delta$  defines the group isomorphism  $\psi_\delta : \Pi_1(y) \rightarrow \Pi_1(\bar{y})$ ,  $[\gamma] \mapsto [\delta^{-1} \cdot \gamma \cdot \delta]$ , where  $\delta^{-1}(t) = \delta(1 - t)$ . Also,  $\delta$  defines the bijection  $\tilde{\psi}_\delta : Q_y \rightarrow Q_{\bar{y}}$ ,  $q \in Q_y \mapsto \bar{q} \in Q_{\bar{y}}$ , such that for the lift  $\tilde{\delta}$  of  $\delta$  to  $Q$  with  $\tilde{\delta}(0) = q$  we have that  $\tilde{\delta}(1) = \bar{q}$ . In addition, the bijection  $\tilde{\psi}_\delta$  is equivariant in sense that  $\tilde{\psi}_\delta([\gamma]q) = \psi_\delta([\gamma])\tilde{\psi}_\delta(q)$ .

Therefore  $\tilde{\delta}$  induces a bijection  $\alpha_\delta : \mathcal{O}(y) \rightarrow \mathcal{O}(\bar{y})$ ,  $O_q \mapsto O_{\tilde{\psi}_\delta(q)}$ , where  $O_q$  is the  $\Pi_1(y)$ -orbit of the point  $q \in Q_y$  and  $O_{\tilde{\psi}_\delta(q)}$  is the  $\Pi_1(\bar{y})$ -orbit of the point  $\tilde{\psi}_\delta(q) \in Q_{\bar{y}}$ .

Let us prove that the loop  $\tilde{\gamma}$  which passes through the points of the orbit  $O_q \in \mathcal{O}(y)$  constructed in Statement 3 is homotopic in  $\pi^{-1}(U'(x))$  to the corresponding loop of the orbit  $O_{\tilde{\psi}_\delta(q)} \in \mathcal{O}(\bar{y})$ .

Let  $\gamma$  be a loop at  $y \in U'(x)$  which represents the positive generator of  $\Pi_1(y)$ . The loop  $\tilde{\gamma}$  constructed in Statement 3 is homotopic to the lift of the loop  $\gamma^k$  starting at a point  $q \in Q_y$ . As the loop  $\gamma^k$  is freely homotopic to the loop  $\bar{\gamma}^k$ , where  $\bar{\gamma} = \delta^{-1}\gamma\delta$ , the lift  $\tilde{\gamma}$  is freely homotopic to the lift of  $\bar{\gamma}^k$  starting at the point  $\tilde{\psi}_\delta(q)$ , but this lift is in turn homotopic to the loop  $\tilde{\gamma}$ .

Therefore the loops  $p\tilde{\gamma}$  and  $p\tilde{\gamma}$  are freely homotopic in  $F$ , therefore define the same element in  $\pi_1(F)$ . Thus we have that  $\text{ind}_x(Q; y, O) = \text{ind}_x(Q; \bar{y}, \alpha_\delta(O))$  for all  $O \in \mathcal{O}(y)$ , hence  $\text{ind}_x(Q; y) = \text{ind}_x(Q; \bar{y})$ .  $\square$

**Corollary 1.** *The set  $\text{ind}_x(Q)$  does not depend on the disk neighborhood  $U(x)$ , this means if  $U_1(x)$  and  $U_2(x)$  are disk neighborhoods of an isolated singular point  $x \in \Sigma$ , and  $y_1 \in U'_1(x)$  and  $y_2 \in U'_2(x)$ , then the set  $\text{ind}_x(Q; y_1)$  constructed via  $U_1(x)$  and the set  $\text{ind}_x(Q; y_2)$  constructed via  $U_2(x)$  coincide.*

*Proof.* Follows from Statement 6  $\square$

From Statement 5 it follows that we can give the following definition.

**Definition 2.** Let  $Q$  be a branched section of the bundle  $\xi$ . The *index* of  $Q$  at  $x \in M$  is

$$(5) \quad \text{ind}_x(Q) = \text{ind}_x(Q; y),$$

where  $y$  is a point of  $U'(x)$ , where  $U(x)$  is a disk neighborhood of  $x$ .

Let us fix an element  $a \in H^1(F)$ . The index of  $Q$  at a point  $x$  with respect to  $a$  is

$$(6) \quad \text{ind}_x(Q; a) = \sum_{O \in \mathcal{O}(y)} \frac{1}{\#O} \langle a, \text{ind}_x(Q; y, O) \rangle = \sum_{O \in \mathcal{O}(y)} \frac{1}{\#O} \int_{\gamma(Q; y, O)} \alpha,$$

where  $\alpha \in \Omega^1(F)$  represents  $a \in H^1(F)$  and  $\gamma(Q; y, O)$  represents the class  $\text{ind}_x(Q; y, O) \in \pi_1(F)$ .

*Example 5.* Let  $M$  be a connected compact oriented manifold and let  $\omega$  be a symmetric tensor of order  $n$  over  $M$ . In Example 2 we have constructed a branched section  $Q \subset PTM$  determined by the binary differential equation (1).

If we consider the covering  $q : \mathbb{S}^1 TM \rightarrow PTM$  given by  $q((p, \vec{v})) = [\vec{v}]$ , we see that  $q \circ \pi : \mathbb{S}^1 TM \rightarrow M$  is a fiber bundle and  $q^{-1}(Q)$  is a  $2n$ -sheeted branched covering of the bundle  $\mathbb{S}^1 TM \rightarrow M$ . Let

$p \in \Sigma$  be a singular point,  $U'(p)$  be a neighborhood disk of  $p$ , and  $\mathcal{O}_p = \{O_1, \dots, O_r\}$  the set of the orbits of the action of  $\pi_1(U'(p))$  on  $\pi^{-1}(p)$ . From the equation (6) it follows that the index of  $q^{-1}(Q)$  at the singular point  $p \in \Sigma$  with respect the cohomology class  $a = [\frac{1}{2\pi}d\theta] \in H^1(\mathbb{S}^1)$ , where  $d\theta$  is the angular form on  $\mathbb{S}^1$  is given by

$$(7) \quad \text{ind}_p(Q; a) = \sum_{i=1}^r \frac{1}{2\pi k_i} \int_{\gamma_i} d\theta,$$

where  $k_i$  is the number of elements of the orbit  $O_i$ , and  $\gamma_i$  is the index of the point  $p$  corresponding to the orbit  $O_i$ . Let us choose a frame  $(e_1, e_2)$  along the curve  $\gamma$ , and we consider a unit vector field  $X(t), 0 \leq t \leq 1$  such that  $\omega_{\gamma(t)}(X(t))$  around the curve  $\gamma : I \rightarrow U'(p)$ . If  $\tilde{\theta}$  is the angle between  $e_1$  and  $X(0)$ , we obtain that the index of  $Q$  at the point  $p$  with respect to the form  $a$  can be also calculated in terms of this rotation angle by the formula

$$(8) \quad \text{ind}_p(Q, O_i, a) = \frac{\tilde{\theta}(2k_i) - \tilde{\theta}(0)}{2\pi k_i}.$$

Note that if the action of  $\pi_1(U'(p))$  on  $\pi^{-1}(p)$  is transitive, then the equation (7) reduces to the following

$$(9) \quad \text{ind}_p(Q; a) = \frac{1}{4\pi n} \int_{\gamma} d\theta,$$

where  $\gamma$  is the index of  $p$  in  $\pi^{-1}(p)$ , and it is also true that

$$(10) \quad \text{ind}_p(Q, \pi^{-1}(p), a) = \frac{\tilde{\theta}(2k_i) - \tilde{\theta}(0)}{4n\pi}.$$

The equation (10) coincides with the index of a binary differential  $n$ -form given in [3].

Now, we note that the index of  $Q$  at a singular point  $x$  seen as a singularity of the bundle  $\pi : PTM \rightarrow M$  is twice the index of the same point as a singular point of the bundle  $\pi \circ q : \mathbb{S}^1 TM \rightarrow M$ .

*Remark 1.* This construction can be used to calculate an index of singular points of singular distributions over a two dimensional manifold  $M$ . In [[5], pages 218-223], the author gives another constructions of indexes of singular points of 1-dimensional singular distributions and branched covering of two sheets defined by such a distributions.

### 3. RESOLUTION OF A BRANCHED SECTION

Let  $M$  be a two-dimensional oriented manifold, and  $\xi = \{\pi_E : E \rightarrow M\}$  be a fiber bundle. Let  $\Sigma$  be a discrete subset of the manifold  $M$ .

**Definition 3.** Let  $Q$  be an  $n$ -sheeted branched section of the bundle  $\xi$  with singularity set  $\Sigma$ ,  $M' = M \setminus \Sigma$ ,  $E' = \pi^{-1}(M')$ , and  $Q' = Q \cap E'$ . A *resolution* of  $Q$  is a map  $\iota : S \rightarrow E$ , where  $S$  is an oriented two-dimensional manifold with boundary, such that

- (1)  $\iota(S) = Q$ ;
- (2)  $\pi = \pi_E \circ \iota : S \rightarrow M$  is surjective;
- (3) the map  $\iota$  is an embedding of  $S' = S \setminus \partial S$  onto  $Q'$ .

In case  $M$  is compact, we assume  $S$  to be compact, too.

*Remark 2.* From Definition 3 it follows that  $\pi_E(Q) = M$  and  $\pi_E(\partial S) = \Sigma$ .

*Example 6.* Let  $M = \mathbb{R}^2$ ,  $E = \mathbb{P}T\mathbb{R}^2$  and a branched section is the solution of the differential equation  $xydx^2 - (x^2 - y^2)dxdy - xydy^2 = 0$ . As the discriminant of this equation is  $(x^2 - y^2)^2 - 4(xy)^2 = (x^2 + y^2)^2$ , this differential equation is a binary differential equation (see Example 2). This differential equation is represented in the form  $(xdx + ydy)(ydx - xdy) = 0$ , therefore its solution  $Q$  consists of two 1-dimensional distributions  $L_1$  and  $L_2$  on  $\mathbb{R}^2$  given respectively by the equations  $xdx + ydy = 0$  and  $ydx - xdy = 0$ . One can easily see that these equations determine sections with singularities  $s_1$  and  $s_2$  of the bundle  $E$ , which admit resolutions (see [1]), call them  $S_1$  and  $S_2$ , so the manifold  $S_1 \sqcup S_2$  is a resolution of the branched section  $Q$ .

*Example 7.* Let  $M = \mathbb{R}^2$ ,  $E = PT\mathbb{R}^2$  and the branched section  $Q$  is the solution of the binary differential equation

$$(11) \quad ydx^2 - 2xdxdy - ydy^2 = 0.$$

The discriminant of equation (11) is  $4(x^2 + y^2)$ , therefore this equation has two real roots for all  $(x, y)$  different from the origin, and at the origin all the coefficients vanish. That is why, equation (11) is a binary differential equation (see Example 2).

The standard coordinates  $(x, y)$  on  $\mathbb{R}^2$  induce a trivialization of the bundle  $\pi_E = E = PT\mathbb{R}^2 \rightarrow M = \mathbb{R}^2$ , namely for the one-dimensional subspace  $l \in PT_{(x,y)}\mathbb{R}^2$  spanned by a vector  $p\partial_x + q\partial_y$ , we assign the point  $(x, y, [p : q]) \in \mathbb{R}^2 \times \mathbb{R}P^1$ . Thus,  $PT\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}P^1$ , and

$$(12) \quad Q = \{(x, y, [p : q]) \in \mathbb{R}^2 \times \mathbb{R}P^1 \mid yp^2 - 2xpq - yq^2 = 0\}$$

In this case

$$(13) \quad \Sigma = (0, 0), \quad Q' = \{(x, y, [p : q]) \in Q \mid x^2 + y^2 > 0\}, \quad M' = \mathbb{R}^2 \setminus \{(0, 0)\}.$$

The projection  $\pi_{PT\mathbb{R}^2} : PT\mathbb{R}^2 \rightarrow \mathbb{R}^2$  restricted to  $Q'$  is a trivial (as a fiber bundle) double covering of  $M'$ . Indeed, take the following open sets  $U_1$  and  $U_2$ :

$$(14) \quad U_1 = M' \setminus (-\infty, 0) \times \{0\} \text{ and } U_2 = M' \setminus (0, \infty) \times \{0\}$$

It is clear that  $M' = U_1 \cup U_2$ . Also, at the points of  $U_1$  we have  $x + \sqrt{x^2 + y^2} > 0$ , and at the points of  $U_2$  we have  $x - \sqrt{x^2 + y^2} > 0$ .

Now let us take two sections of the bundle  $\pi_{PT\mathbb{R}^2} : PT\mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined on  $M'$ :

$$(15) \quad s_1 : (x, y) \mapsto \begin{cases} (x, y, [x + \sqrt{x^2 + y^2} : y]), & (x, y) \in U_1, \\ (x, y, [-y : x - \sqrt{x^2 + y^2}]), & (x, y) \in U_2, \end{cases}$$

and

$$(16) \quad s_2 : (x, y) \mapsto \begin{cases} (x, y, [-y : x + \sqrt{x^2 + y^2}]), & (x, y) \in U_1, \\ (x, y, [x - \sqrt{x^2 + y^2} : y]), & (x, y) \in U_2, \end{cases}$$

Note that over  $U_1 \cap U_2$  there holds

$$(17) \quad [x + \sqrt{x^2 + y^2} : y] = [-y : x - \sqrt{x^2 + y^2}] \text{ and } [-y : x + \sqrt{x^2 + y^2}] = [x - \sqrt{x^2 + y^2} : y],$$

therefore the sections  $s_1$  and  $s_2$  are well defined. One can easily prove that  $s_i(M') \subset Q'$ ,  $i = 1, 2$ , and  $s_1(M') \cap s_2(M') = \emptyset$ . Therefore  $Q'$  is a trivial double covering of  $M'$ .



Now let us construct a resolution of the branched section  $Q$ . Recall that  $\mathbb{S}^1 = \{(u, v) \mid u^2 + v^2 = 1\}$ , then let take the diffeomorphism

$$(18) \quad f : \mathbb{S}^1 \rightarrow \mathbb{R}P^1, (u, v) \mapsto \begin{cases} [u+1 : v], & u > -1, \\ [-v : u-1], & u < -1, \end{cases}$$

and then the diffeomorphism  $f$  “rotated” at the angle  $\pi/2$  gives the diffeomorphism,

$$(19) \quad g : \mathbb{S}^1 \rightarrow \mathbb{R}P^1, (u, v) \mapsto \begin{cases} [-v : u+1], & u > -1, \\ [u-1 : v], & u < -1. \end{cases}$$

We take  $S_1 = S_2 = \mathbb{R}_+ \times \mathbb{S}^1 = [0, \infty) \times \mathbb{S}^1$ , and  $S'_1 = S'_2 = (0, \infty)$ . We set  $S = S_1 \sqcup S_2$ , then  $S' = S'_1 \sqcup S'_2$ . Then  $\iota : S \rightarrow \mathbb{R}^2 \times \mathbb{R}P^1$  is given by

$$(20) \quad \iota|_{S_1}(r, (u, v)) = (ru, rv, f(u, v)), \quad \iota|_{S_2}(r, (x, y)) = (ru, rv, g(u, v)).$$

One can easily see that  $\iota|_{S'_i} : S'_i \rightarrow Q'_i$ ,  $i = 1, 2$  is a diffeomorphism. For example, any point  $(x, y, [p : q]) \in V_{11}$ , is the image of the point  $(r, (u, v))$  under the map  $\iota|_{S_1}$ , where

$$(21) \quad u = \frac{x}{\sqrt{x^2 + y^2}}, \quad v = \frac{y}{\sqrt{x^2 + y^2}}, \quad r = \sqrt{x^2 + y^2}.$$

*Example 8.* As a generalization of Examples 6 and 7 one can take  $n$  sections with singularities [1] of a bundle  $\xi = \pi_E : E \rightarrow M$  which have the same set of singularities  $\Sigma$ , call them  $s_i$ ,  $i = \overline{1, n}$ . These sections define a branched section  $Q$  of the bundle  $\xi$ :  $Q = \{s_i(x) \mid x \in M \setminus \Sigma\}$ . If  $S_i$  is a resolution of  $s_i$ , then  $S = \sqcup S_i$  is a resolution of  $Q$ .

*Example 9.* Let us present an example of branched section, where the covering  $\pi_Q|_{Q'} : Q' \rightarrow M'$  is not trivial. Take  $M = \mathbb{R}^2 = \mathbb{C}$ ,  $E = \mathbb{S}^1(\mathbb{C}) = \mathbb{C} \times \mathbb{S}^1$ , the bundle of unit vectors over  $M$ , and let

$$(22) \quad Q = \{(z, w) \in \mathbb{C} \times \mathbb{S}^1 \mid |z|w^2 = z\}.$$

Then  $M' = \mathbb{C} \setminus \{0\}$ ,  $Q' = \{(z, w) \mid w^2 = z/|z|\}$ , and it is well known that  $\pi_Q|_{Q'} : Q' \rightarrow M'$  is a non trivial double covering. Now let us take

$$(23) \quad S = [0, \infty) \times \mathbb{S}^1, \text{ and } \iota : S \rightarrow E, \quad (r, e^{i\varphi}) \mapsto (re^{2i\varphi}, e^{i\varphi})$$

Then  $S' = (0, \infty)$ , and it is clear that the properties (1)–(3) of Definition 3 hold true for  $\iota$ .

*Example 10.* Let us present another example of branched section, where the covering  $\pi_Q|_{Q'} : Q' \rightarrow M'$  is not trivial. Take  $M = \mathbb{R}^2 = \mathbb{C}$ ,  $E = PT\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}P^1 = \mathbb{C} \times \mathbb{R}P^1$ , and let

$$(24) \quad Q = \{(z, [w]) \mid |w| = 1 \text{ and } |z|^2 w^4 = z^2\}$$

Then  $M' = \mathbb{C} \setminus \{0\}$ ,  $Q' = \{(z, w) \mid w^4 = z^2/|z|^2\}$ , and it is clear that  $\pi_Q|_{Q'} : Q' \rightarrow M'$  is a non trivial double covering. Now let us take

$$(25) \quad S = [0, \infty) \times \mathbb{R}P^1, \text{ and } \iota : S \rightarrow E, \quad (r, [w]) \mapsto (rw^2, [w]),$$

where  $|w| = 1$ . Then  $S' = (0, \infty)$ , and it is clear that the properties (1)–(3) of Definition 3 hold true for  $\iota$ .

*Remark 3.* In Examples 9–10, for each  $x \in M$ , the set  $S_x$  is a discrete set if  $x \in M \setminus \Sigma$ , or is diffeomorphic to a circle  $\mathbb{S}^1$  if  $x \in \Sigma$ .

Now let us consider a point  $x \in \Sigma$ . Then, according to Definition 3,  $S_x = \pi^{-1}(x)$  consists of the connected components of the boundary  $\partial S$ . Let us denote by  $C(S_x)$  the set of connected components of  $S_x$ . As  $S_x$  is compact, the set  $C(S_x)$  is finite, and each element of this set is diffeomorphic to a circle  $\mathbb{S}^1$ .

**Statement 6.** *Let  $C$  be a connected component of a boundary. Then there exists a neighborhood  $N(C)$  of  $C$  and a diffeomorphism  $f_C : N(C) \rightarrow \mathbb{S}^1 \times [0, 1]$  such that  $f_C(C) = \mathbb{S}^1 \times \{0\}$  and  $U(x) = \pi(N(C))$  is a disk neighborhood of  $x$ . For each  $y \in U'(x)$ , the set of orbits  $\mathcal{O}_y$  consists of only one element. In this cases the curve  $\hat{\gamma}$  corresponding to the orbit by Statement 3 is a generator of the group  $\pi_1(N(C)) \cong \mathbb{Z}$ .*

*Proof.* Indeed,  $N(C) \setminus C$  is homeomorphic to a ring and  $U'(x)$  is homeomorphic to a ring as well. The map  $N(C) \setminus C \rightarrow U'(x)$  induced by  $\pi$  is a  $n$ -fold covering therefore  $\pi_* : \pi_1(N(C)) \cong \mathbb{Z} \rightarrow \pi_1(U'(x))$  has the form  $m \rightarrow km$ . At the same time  $\pi_*([\tilde{\gamma}]) = \gamma^k$ , thus  $[\tilde{\gamma}]$  is a generator of the group  $\pi_1(N(C))$ .  $\square$

**Corollary 2.** *The curve  $\tilde{\gamma}$  is homotopic in  $N(C) \subset S$  to the curve  $C \subset E_x$ . Therefore the curve  $C$  represents  $\text{ind}_x(Q, O)$ .*

#### 4. CONNECTION AND THE GAUSS-BONNET THEOREM

Let  $\xi = (\pi_E : E \rightarrow M)$  be a locally trivial fiber bundle with standard fiber  $F$  and structure group  $G$ . Assume that  $G$  is a connected Lie group.

Let  $(U, \psi : \pi^{-1}(U) \rightarrow U \times F)$  be a chart of the atlas of  $\xi$ . Let

$$(26) \quad \eta = p_F \circ \psi : \pi^{-1}(U) \rightarrow F,$$

where  $p_F : U \times F \rightarrow F$  is the canonical projection onto  $F$ . For each  $x \in U$  the map  $\eta$  restricted to  $F_x = \pi^{-1}(x)$  induces a diffeomorphism  $\eta_x : F_x \rightarrow F$ , and let  $i_x : F \rightarrow F_x$  be the inverse of  $\eta_x$ . Note that if we take another chart  $(U', \psi' : \pi^{-1}(U') \rightarrow U' \times F)$ , and  $\eta' : \pi^{-1}(U') \rightarrow F$  is the corresponding map, then on  $\pi^{-1}(U \cap U')$  we have that

$$(27) \quad \psi' \circ \psi^{-1} : (U \cap U') \times F \rightarrow (U \cap U') \times F, \quad (x, y) \mapsto (x, g(x)y),$$

where  $g : U \cap U' \rightarrow G$  is the gluing map of the charts. Now, for any  $x \in U \cap U'$ , we have  $\eta'_x \circ \eta_x^{-1}(y) = g(x)y$ , and, as  $G$  is connected,  $\eta'_x \circ \eta_x^{-1} : F \rightarrow F$  is homotopic to the identity map. This means that for any  $x \in m$  we have well defined isomorphisms of the homotopy and (co)homology groups:

$$(28) \quad \begin{aligned} \pi_*(\eta_x) : \pi_*(F_x) &\rightarrow \pi_*(F), \\ H_*(\eta_x) : H_*(F_x) &\rightarrow H_*(F), \quad H^*(\eta_x) : H^*(F) \rightarrow H^*(F_x), \end{aligned}$$

which do not depend on the chart.

In [1], for a locally trivial bundle with standard fiber  $F$  and structure Lie group  $G$ , we have proved the following statement ([1], Statement 1):

**Statement 7.** *Let  $a \in H^1(F)$  and  $H$  be a connection in  $E$ . There exists a 1-form  $\alpha \in \Omega^1(E)$  such that*

- (1)  $\alpha|_H = 0$ ;
- (2) for each  $x \in M$ ,  $di_x^* \alpha = 0$  and  $[i_x^* \alpha] = H^1(\eta_x)a$ .

The decomposition  $TE = H \oplus V$  gives a bicomplex representation of the complex  $\Omega(E)$ , then the form  $\alpha$  lies in  $\Omega^{(0,1)}(E)$  and  $d\alpha = \theta_{(1,1)} + \theta_{(2,0)}$ , where  $\theta_{(1,1)} \in \Omega^{(1,1)}$  and  $\theta_{(2,0)} \in \Omega^{(2,0)}$ , and

$$(29) \quad \theta_{(1,1)}(X, Y) = (L_X \alpha)(Y), \quad \theta_{(2,0)} = \tilde{\alpha}(\Omega).$$

where  $L_X$  is the Lie derivative with respect to the vector field  $X$ , and  $\Omega$  is the curvature form of the connection  $H$  (for details see [1], Section 3).

Now let  $Q$  be a branched section of the bundle  $\xi$  which admits a resolution  $\iota : S \rightarrow E$  (see Definition 3). Let us fix an element  $a \in H^1(F)$ , and let  $\alpha \in \Omega^1(E)$  be the corresponding 1-form (see Statement 7). Then, by the Stokes theorem we have

$$(30) \quad \int_{\partial S} \iota^* \alpha = \int_S \iota^* d\alpha.$$

By Remark 2 we have that  $\pi_E(\partial S) = \Sigma$ . For  $x \in \Sigma$ , let  $C(S_x)$  be the set of connected components of  $\pi_E^{-1}(x)$ .

From Corollary 2, it follows that, for  $C \in C(S_x)$ , we have

$$(31) \quad \int_C \alpha = \int_{\gamma(Q; y, O(C))} i_x^* \alpha,$$

where  $\gamma(Q; y, O(C))$  represents the class  $ind_x(Q; y, O(C)) \in \pi_1(F)$ , and  $O(C)$  is the orbit of the local monodromy group corresponding to  $C$ . Therefore, from (6) we have that

$$(32) \quad ind_x(Q; a) = \sum_{C \in C(S_x)} \frac{1}{\#O(C)} \int_C \alpha.$$

If all the orbits of the local monodromy group corresponding to the components  $C \in C(S_x)$  have the same number of elements  $N(x)$ , then

$$(33) \quad \int_{\partial S} \iota^* \alpha = \sum_{x \in \Sigma} \sum_{C \in C(S_x)} \int_C \alpha = \sum_{x \in \Sigma} N(x) ind_x(Q; a)$$

Thus we get the following theorem

**Theorem 1** (Gauss-Bonnet-Hopf-Poincaré for branched sections). *If, for any  $x \in \Sigma$ , all the orbits of the local monodromy group corresponding to the components  $C \in C(S_x)$  have the same number of elements  $N(x)$ , then*

$$\int_S \iota^* \theta_{(1,1)} + \iota^* \theta_{(2,0)} = \sum_{x \in \pi(\partial S)} N(x) ind_x(Q).$$

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